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TUNING OF MUSICAL NOTES THROUGH MATHEMATICS

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Abstract— Mathematics, an enigma in numbers and calculations, often accompanied by feelings of rejection and disinterest while Music, a flow with emotions, feelings and life. Motivation for investigating the connections between these two apparent opposites' poles is attempted in this paper. A correlation between Mathematics and Music is shown.. Music theorists sometimes use mathematics to understand music. Mathematics is "the basis of sound" and sounds itself "in its musical aspects... exhibits a remarkable array of number properties", simply because nature itself "is amazingly mathematical". In today's technology, without mathematics it is difficult to imagine anything feasible. In this paper we have discussed the relation between music and mathematics. How piano keys are interrelated with mathematics, frequencies are correlated and discussed. With the aid of mathematical tools, regression, geometric progression, tuning frequency can be calculated and further related to key of piano used to produce that particular frequency . This paper will also be helpful for music seekers and mathematician to understand easily a relationship between Mathematics and Music from a mathematician prospective.[12]

Index Terms— Piano frequencies, regression, geometric progression,piano keyboard, tuning, sine wave.

1 Introduction

The patterns that exist between math, language, and music have prompted numerous studies to be commissioned to establish their inter- relationship. Music is a series of notes that are played in accordance to a pattern and math's too works in a similar way. In math's a result always remains finite despite the various ways in which you can add, multiply, subtract, and divide numbers.

The same can be said about music. Notes can be combined in an endless variety of groupings but the number of notes and sounds that exist are finite. It is these patterns and combinations that make music and math very similar.[11].

Mathematics and music are interconnected topics. "Music gives beauty and another dimension to mathematics by giving life and emotion to the numbers and patterns." Mathematical concepts and equations are connected to the designs and shapes of musical instruments, scale intervals and musical compositions, and the various properties of sound and sound production. This paper will allow exploring several aspects of mathematics related to musical concepts.

The ancient Greeks figured out that the integers correspond to musical notes. Any vibrating object makes overtones or harmonics, which are a series of notes that emerge from a single vibrating object. These notes form the harmonic series: 1/2, 1/3, 1/4, 1/5 etc. The fundamental musical concept is probably that of the octave. A musical note is a vibration of something, and if you double the number of vibrations, you get a note an octave higher; likewise if you halve the number of vibrations, it is an octave lower. Two notes are called an interval; three or more notes is a chord. The octave is an interval common to all music in the world. Many people cannot even distinguish between notes an octave apart, and hear them as the same.

A musical keyboard is the set of adjacent depressible levers or keys on a musical instrument, particularly the piano. Keyboards typically contain keys for playing the twelve notes of the Western musical scale, with a combination of larger, longer keys and smaller, shorter keys that repeats at the interval of an octave. Depressing a key on the keyboard causes the instrument to produce sounds, either by mechanically striking a string or tine (piano, electric piano, clavichord); plucking a string (harpsichord); causing air to flow through a pipe (organ); or strike a bell (carillon). On electric and electronic keyboards, depressing a key connects a circuit (Hammond organ, digital piano, and synthesizer). Since the most commonly encountered keyboard instrument is the piano, the keyboard layout is often referred to as the "piano keyboard".

The twelve notes of the Western musical scale are laid out with the lowest note on the left; The longer keys (for the seven "natural" notes of the C major scale: C, D, E, F, G, A, B) jut forward. Because these keys were traditionally covered in ivory they are often called the white notes or white keys. The keys for the remaining five notes—which are not part of the C major scale—(i.e .,C♯, D♯, F♯, G♯, A♯) are raised and shorter. Because these keys receive less wear, they are often made of black colored wood and called the black notes or black keys. The pattern repeats at the interval of an octave.

1.1 Piano Keyboard [10]

For understanding of this paper, it is important to have some knowledge of the piano keyboard, which is illustrated in the following diagram. This keyboard has 88 keys of which 36 (the top of the illustration), striking each successive key produces a pitch with a particular frequency that is higher than the pitch produced by striking the previous key by a fixed interval called a semitone. The frequencies increase from left to right. Some examples of the names of the keys are A0, A0#, B0, C1, C1#. For the purposes of this paper, all the black keys will be referred to as sharps (#). In this paper different frequencies of piano are discussed, how they are produced periodically with the use of Regression Analysis and Geometric Progression.

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Diagram illustrates different key numbers, key names and their corresponding frequencies in piano keyboard. From key numbers 1 to 12 frequencies are given, but from 13 to 24 they form the same pattern but double the initial values and from 25 to 36 values are thrice of initial values and so on.

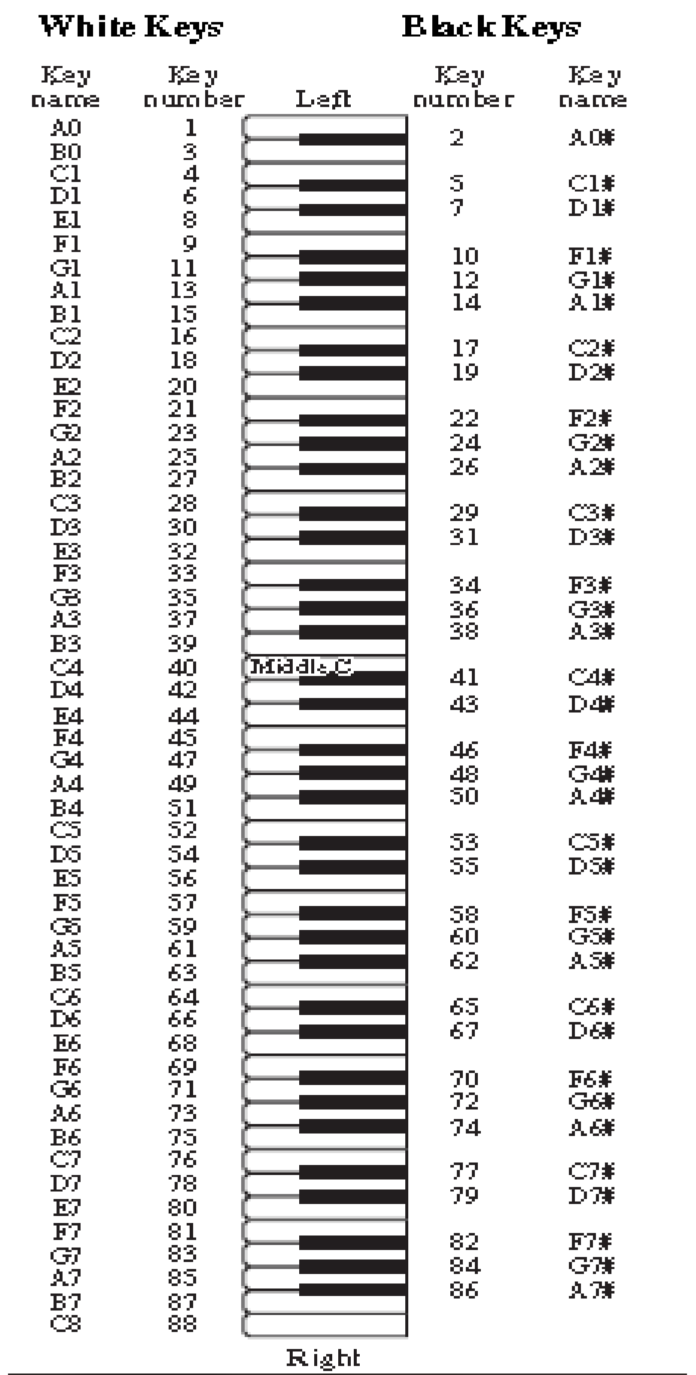
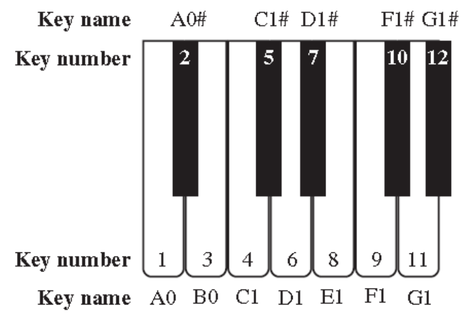


Fig. 1 ***Piano Keyboard[10]***

The frequencies of all successive pitches produced by striking the keys on a piano keyboard form a pattern. The diagram on the left shows the first 12 keys of a piano. The table down shows the frequency of the pitch produced by each key, to the nearest thousandth of a Hertz (Hz).



Key Name	Key Number	Frequency (Hz)
A0	1	27.500
A0#	2	29.135
B0	3	30.868
C1	4	32.703
C1#	5	34.648
D1	6	36.708
D1#	7	38.891
E1	8	41.203
F1	9	43.654
F1#	10	46.249
G1	11	48.999
G1#	12	51.913

Fig. 2 Different Frequencies on Piano Keyboard

2 REGRESSION

In statistics, regression analysis includes many techniques for modeling and analyzing several variables, when the focus is on the relationship between a dependent variable and one or more independent variables. More specifically, regression analysis helps one understand how the typical value of the dependent variable changes when any one of the independent variables is varied, while the other independent variables are held fixed. Regression is of two types,
Linear Regression and Exponential Regression
A linear regression produces the slope of a line that best fits a

single set of data points. For example a linear regression could be used to help project the sales for next year based on the sales from this year.
An exponential regression produces an exponential curve that best fits a single set of data points. For example an exponential regression could be used to represent the growth of a population. This would be a better representation than using a linear regression.
Best fit associated with n points

$$(x_1, y_1), (x_{2,} y_2), ... (x_{n,} y_n)$$

has exponential formula,

$$y = ar^x$$

Taking log both sides

$$\log y = \log a + x \log r$$

Equating with Y= mx + b
Slope $m = \log r$

Intercept $b = \log a$

Best fit line using log y as a function of x.

$$r = 10^m$$

$$a = 10^b$$

Equating above table with Regression analysis, taking key number as x and frequencies as y = log (frequency).

Equations for exponential regression

Slope =

$$m = \frac{n\Sigma(xy) - (\Sigma x)(\Sigma y)}{n\Sigma x^2 - (\Sigma x)^2}$$

$$m = \frac{12 \times 126.61342 - (78) \times (18.92718)}{12 \times 650 - (78)^2}$$

Key Name	Key Number	Frequency (Hz)
A0	1	27.500
A0#	2	29.135
B0	3	30.868
C1	4	32.703
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D1#	7	38.891
E1	8	41.203
F1	9	43.654
F1#	10	46.249
G1	11	48.999
G1#	12	51.913

$$m = .0250821$$

Intercept =

$$b = \frac{(\Sigma y) - m(\Sigma x)}{n}$$

$$b = 1.41423135$$

$$y = mx + b$$

$$y = .0250821x + 1.41423135$$

3 Geometric Progression

The frequencies of pitches produced by striking the piano keys can also be modeled by a geometric sequence. The model can be determined by using a pair of keys with the same letter and consecutive numbers; for example, A0 and A1, or B1 and B2, or G2# and G3#. Each pair of consecutive keys with the same letter has frequencies with a ratio of 2:1. In other words, the frequency of A1 (55.000 Hz) is double the frequency of A0 (27.500 Hz), the frequency of A2 (110.000 Hz) is double the frequency of A1 (55.000 Hz), and so on. In mathematics, a geometric progression, also known as a geometric sequence, is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed non-zero number called the common ratio. For example, the sequence 2, 6, 18, 54 ... is a geometric progression with common ratio 3. Similarly 10, 5, 2.5, 1.25, is a geometric sequence with common ratio 1/2. The sum of the terms of a geometric progression, or of an initial segment of a geometric progression, is known as a geometric series. Thus, the general form of a geometric sequence is

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots$$

$$a_0, a_1, a_2 .... a_{n-1}$$

Or

$$a, a_1, a_2 .... a_{n-1}$$

nth term will be

$$a_n = ar^{n-1}$$

Where a is the first term and r is the common ratio.

$$r = \frac{a_1}{a}$$

Referring table no. 2.1
So if we take frequencies of key numbers in geometric progression then first term will be 27.500 and second term will be 29.135, so r = 29.135/27.500 = 1.05945

$$a_n = ar^{n-1}$$

$$a_n = 27.500(1.05945)^{n-1}$$

4 Comparison of Frequencies through Regression Analysis and Geometric Progression.

TABLE 1
COMPARISON OF FREQUENCIES

K.N	Frequency (Actual) Hz	Frequency (Regression) Hz	Frequency (GP) Hz	K.N	Frequency (Actual) Hz	Frequency (Regression) Hz	Frequency (GP) Hz
A0	27.500	27.49878	27.500	A1	2*27.500= 55.000	54.99189	54.99184
A0#	29.135	29.13369	29.13487	A1#	2*29.135= 58.270	58.26137	58.26110
B0	30.868	30.86580	30.86700	B1	2*30.868= 61.736	61.725249	61.72473
C1	32.703	32.70090	32.70090	C2	2*32.703= 65.406	65.39511	65.39426
C1#	34.648	34.645102	34.64611	C2#	2*34.648= 69.296	69.28306	69.28196
D1	36.708	36.704891	36.70583	D2	2*36.708= 73.416	73.40221	73.40077
D1#	38.891	38.88714	38.88798	D2#	2*38.891= 77.782	77.76627	77.76444
E1	41.203	41.19914	41.19988	E2	2*41.203= 82.406	82.10991	82.38754
F1	43.654	43.64859	43.64921	F2	2*43.654= 87.308	87.2882	87.28548
F1#	46.249	46.24367	46.24416	F2#	2*46.249= 92.498	92.477821	92.47460
G1	48.999	48.99305	48.99337	G2	2*48.999= 97.998	97.97599	97.97222
G1#	51.913	51.90588	51.90603	G2#	2*51.913= 103.826	103.80135	103.79666

KN = Key Name, GP = Geometric Progression

As we can observe from the table, moving from key numbers 1 to 12, Geometric progression is more effective in determining values of frequencies near to actual frequencies. But as we proceed further towards 13, onwards to higher numbers, Regression Analysis is the method to count upon in determining frequencies quite close to actual frequencies.

If a single key produces a frequency of 783.5Hz, than which is this key.
From Regression analysis y = .0250821x + 1.41423135

$$10^y = f = 783.5$$

After solving x= 58.99.

From geometric progression analysis

$$a_n = 27.500(1.05945)^{n-1}$$
$$783.5 = 27.500(1.05945)^{n-1}$$

After solving n= 59.00
So 59 = 12x5 -1 = equal to 11 = G1 Key
Or 783.5 = 48.99(G1) x16 =783.9 = a multiple of frequency of key G1.
So any frequencies produced by the piano can be related to given key.

5 TUNING

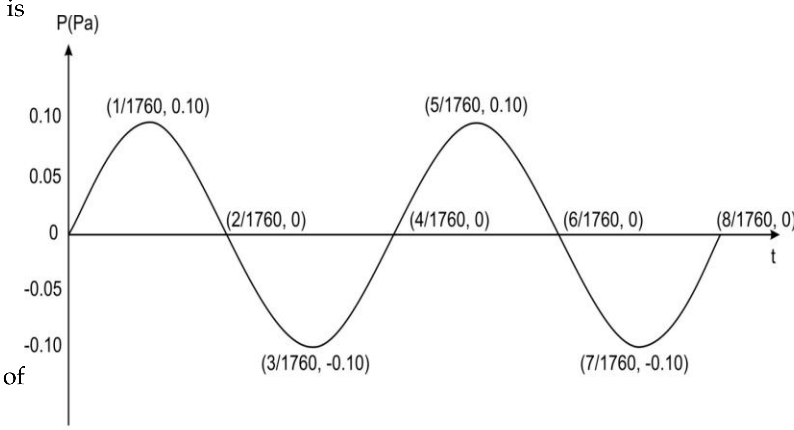
Tuning is the process of adjusting the pitch of one or many tones from musical instruments to establish typical intervals between these tones. Tuning is usually based on a fixed reference, such as A = 440 Hz. Out of tune refers to a pitch/tone that is either too high (sharp) or too low (flat) in relation to a given reference pitch. While an instrument might be in tune relative to its own range of notes, it may not be considered 'in tune' if it does not match A = 440 Hz (or whatever reference pitch one might be using). Some instruments become 'out of tune' with damage or time and have to be readjusted or repaired. Some instruments produce a sound which contains irregular overtones in the harmonic series, and are known as inharmonic. Tuning may be done aurally by sounding two pitches and adjusting one of them to match or relate to the other. A tuning fork or electronic tuning device may be used as a reference pitch, though in ensemble rehearsals often a piano is used (as its pitch cannot be adjusted for each rehearsal).
When tuning a musical instrument such as a piano, a violin, or a guitar, a tuning fork or an electronic tuning device is used to produce a pure pitch at a specific frequency.
The tuning fork used most often is for the key A4 (f = 440 Hz) and is referred to as an A440 tuning fork. A tuning fork produces a sound wave whose pressure variations
can be modelled by a sinusoidal function of the form.

$$P = a\sin(2\pi ft)$$

Swhere P = pressure variation of the sound wave in pascals (Pa)
a = amplitude of the sound wave in pascals
f = frequency of the sound wave in Hertz (Hz)
t = time in seconds
A violin is being tuned with an A440 tuning fork (f = 440 Hz). The sound wave that is produced by the tuning fork has an amplitude of 0.10 Pa. This sound wave can be represented by the function

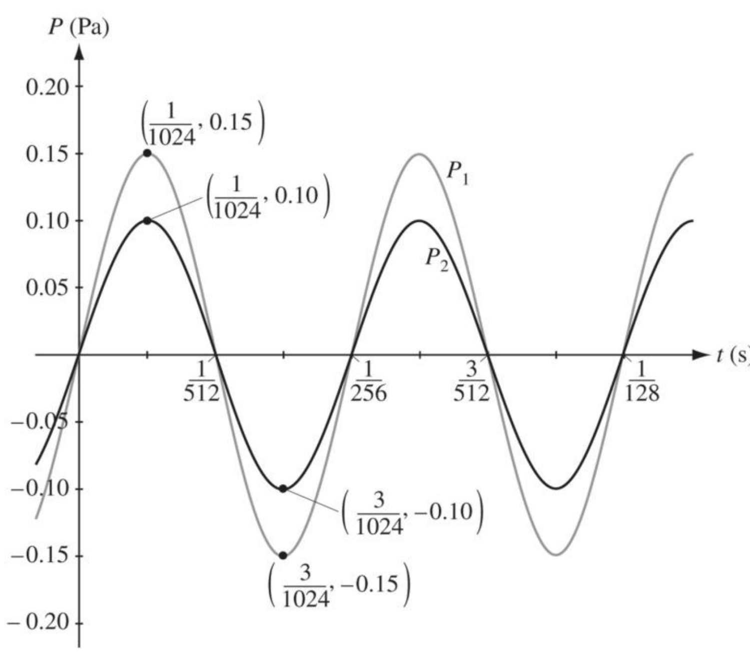
$$P = 0.10\sin(880\pi t)$$

Plotting P(Pa) on y axis and taking t on x axis, following graph is obtained.



P = 0.10 Sin(2π x 440t)

Another tuning fork with a different frequency is being used to tune a particular pitch (key) on a piano. The tuning fork is struck twice, with a different force each time. A graph representing each of the sound waves produced is shown on the coordinate plane below.



From the above graph ,when t= 1/1024 then P= 0.15Pa and when t = 3/1024 then P= -0.15Pa

So as $P = a\sin(2\pi ft)$

$$0.15Pa = a\sin(2\pi f \times \frac{1}{1024}) \quad \text{------------(i)}$$

$$-0.15Pa = a\sin(2\pi f \times \frac{3}{1024}) \quad \text{------------(ii)}$$

Solving (i) and (ii)

$$2\pi f \times \frac{3}{1024} = \pi + 2\pi f \times \frac{1}{1024}$$

$$f = 256Hz$$

So tuning fork is of frequency 256Hz, and the key of piano which has been tuned is,

From Regression analysis

y = .0250821x + 1.41423135

$10^y = f = 256$

After solving x= 39.63

From Geometric progression

$$a_n = 27.500(1.05945)^{n-1}$$

$$256 = 27.5(1.05945)^{n-1}$$

Solving n = 39.63
So 39.63= 12x3 +4 = equal to 4 = C4 Key
Or 256 = 32.7009(C1) x8 =256 = a multiple of frequency of key C1
So any frequencies produced by the piano can be related to given key.

6 Conclusion and Further work

In this paper we have elaborated the fact that music and mathematics are interrelated. The different frequencies used in music are based on mathematical calculations. Paper discusses two cases, first where a frequency was known and another with unknown frequency. After calculating frequency through graph, the required key in piano is found with the help of regression and geometrical progression.. Both the methods are effective and have produced desired results.. Further work in determining frequencies and their pattern can be done through Fourier Transform. This paper will help both music seekers as well as mathematical intellectuals a belief that both mathematics and music are interconnected.Tunning frequency can be calculated, with further finding of piano key which is producing the desired result.

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